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Finite Temperature Field Theory and
Quantum Noise in an Electrical Network

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ABSTRACT

Finite temperature field (FTF) theory is used to study quantum noise in an electrical network. Numerical solutions for the finite second moments which satisfy the uncertainty principle bound are given for a dissipative quantum oscillator.

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The understanding of quantum and thermal noise in electrical networks is essential for the development of sensitive electrical instruments of the type used to detect and transmit weak signals in communication systems, computers, and instruments designed to detect gravitational radiation. In this work, it is demonstrated how FTF theory leads to a simple derivation of the fluctuation dissipation theorem and the relation between thermal and quantum noise in an electrical network. When the dissipative elements are represented by semi-infinite filters, a procedure results that yields finite second moments for both charge and current as well as compatibility with the uncertainty bound. This method is demonstrated with an investigation of a series LRC dissipative quantum oscillator, and analytical and numerical results are given which illustrate the features of thermal and quantum noise. Other approaches to the dissipative quantum oscillator are discussed and reviewed in Refs. [1-4].

Here, I follow the notation and quantization procedure developed in Ref. [1]; however, the present approach differs in that FTF's are introduced so as to include temperature dependence in the expressions for quantum noise. In addition, the ideal transmission lines are replaced with semi-infinite low-pass filters composed of reapeated basic elements of inductance ($L_0 = L_T\Delta x$) and capacitance ($C_0 = C_T\Delta x$) as shown in Fig. 1. The characteristic impedance of the filter [5],

$$Z(a,b) = i\omega L_0/2 + (L_0/C_0 - \omega^2 L_0^2/4)^{\frac{1}{2}}, \qquad (1)$$

implies a maximum frequency above which the voltage wave becomes damped. Below this frequency the voltage in the n^{th} element of the filter is

$$V(n) = \exp(ikn\Delta x)V(o)$$

$$k/2 \approx \tan(k/2) = \frac{L_T\,\omega}{2(L_0/C_0 - \omega^2 L_0^2/4)^{\frac{1}{2}}} = \omega/2v(\omega) \qquad (2)$$

relates the phase velocity ($v(\omega) = R(\omega)/L_T$), the wave number k, and the real part $R(\omega)$ of the characteristic impedance (1).

At frequency ω, the Lagrangian density for the LCR network is

$$\mathcal{L}(Q_\omega, \partial_t Q_\omega, \beta) = \frac{\delta(x)}{2}\left[L\left(\frac{\partial}{\partial t} Q_\omega(x,t,\beta)\right)^2 - Q_\omega^2(x,t,\beta)/C\right]$$

$$+ \frac{L_T H(x)}{2}\left[\left(\frac{\partial Q_\omega(x,t,\beta)}{\partial t}\right)^2 - v^2(\omega)\left(\frac{\partial}{\partial x} Q_\omega(x,t,\beta)\right)^2\right] \tag{3}$$

where $\delta(x)$ and $H(x)$ are respectively the Dirac and Heaviside distributions and where $Q(x,t,\beta)$, $\beta = 1/kT$, is the spectral charge density and $\Phi(x,t,\beta)$ its conjugate momentum. The postulates of quantization [6] suggest the commutation relation

$$[Q(x,t,\beta), \Phi(x',t,\beta)] = i\hbar\,\delta(x-x'), \tag{4a}$$

where

$$Q(x,t,\beta) = \int_0^{\omega_{Max}} Q_\omega(x,t,\beta)\,d\omega \tag{4b}$$

A derivation similar to that in Ref. [1] yields the spectral Lagenvin equation

$$\int_0^{\omega_{Max}}\left(L\ddot{Q}_\omega(t,\beta) + R(\omega)\dot{Q}_\omega(t,\beta) + \frac{Q_\omega(t,\beta)}{C} - 2R(\omega)\dot{Q}^{in}_\omega(t,\beta)\right)d\omega = 0 \tag{5}$$

where dissipation is represented by a spectral damping coefficient similar to that of the Drude model [4] and given by

$$\gamma(\omega) = R(\omega)/L = (R/L)\left(1-(\nu/\Lambda)^2\right)^{1/2} \tag{6}$$

where $R = (L_0/C_0)^{\frac{1}{2}}$, $\nu = \omega/\omega_0$, $\Lambda = 2Q_0C/C_0$, $\omega_0 = (LC)^{-\frac{1}{2}}$, and $Q_0 = L\omega_0/R$. Here ω_0 is the natural frequency of the LC system, Q_0 the quality factor, and $\Lambda\omega_0$ the highest frequency passed by the filter.

The normalized Heisenberg FTF (4b) becomes

$$Q(t,\beta) = \left(\frac{\hbar}{2\pi K_2(Q_0,0)}\right)^{1/2}\int_0^{\Lambda\omega_0} d\omega\,(\omega R(\omega))^{1/2}\left\{\frac{i A^{in}(\omega)\, e^{-i\omega t}}{(\omega^2 L - 1/C) + i\omega R(\omega)} + \text{H.c.}\right\} \tag{7}$$

when the boson in-field operator is expressed in terms of the FTF operator [7]

$$A^{in}(\omega) = (1+f(\beta))^{1/2} A(\omega,\beta) + f^{1/2}(\beta)\,\tilde{A}^{\dagger}(\omega,\beta) \tag{8}$$

The matrix element of the number operator $N^{in}(\omega) = A^{\dagger in}(\omega)\, A^{in}(\omega)$ between finite temperature vacuum states is

$$(\beta 0|N^{in}(\omega)|0\beta) = f(\beta) = 1/(\exp(\omega\beta) - 1) \tag{9}$$

The Heisenberg field (7) satisfies

$$[Q(t,\beta), L\dot{Q}(t,\beta)] = i\hbar \tag{10}$$

which implies the uncertainty inequality

$$\sigma(Q,\beta)\,\sigma(L\dot{Q},\beta) \geq \hbar/2 \tag{11}$$

Using (7)-(9), the second moments become the fluctuation-dissipation theorem results [8,9] modified by spectral damping

$$\sigma^2(Q,z) = (\beta 0|Q^2(t,\beta)|0\beta) = \frac{\hbar K_1(Q_0,z)}{L\omega_0 2K_2(Q_0,0)} \tag{12}$$

$$\sigma^2(L\dot{Q},z) = (\beta 0|(L\dot{Q}(t,\beta))^2|0\beta) = \frac{\hbar L\omega_0 K_3(Q_0,z)}{2K_2(Q_0,0)}$$

$$K_m(Q_0,z) = \int_0^\Lambda \frac{\nu^m \coth(\nu/z)\,d\nu}{\pi Q_0(\nu)\left[(\nu^2-1)^2+(\nu/Q_0(\nu))^2\right]} \tag{13}$$

$$Q_0(\nu) = Q_0(1-(\nu/\Lambda)^2)^{-1/2}$$

The results for the zero temperature ($z=2kT/\omega_0=0$) vacuum are found with the replacement $\coth(\nu/z)\longrightarrow 1$.

In the large Q_0 ($R \to 0$) limit, one finds the LC oscillator result from (12) and (13) since $2K_2(Q_0,0)\rightarrow 1$ and $K_1(Q_0,z)\longrightarrow K_3(Q_0,z)\longrightarrow \coth(1/z)/2$. The zero temperature vacuum results are

$$2\pi Q_0 K_3(Q_0,0) = (1/2)\ln\left[(\Lambda^2-1)^2+(\Lambda/Q_0)^2\right] + (1-1/2Q_0^2)2\pi Q_0 K_1(Q_0,0) \tag{14}$$

$$2\pi Q_0 K_1(Q_0,0) = \frac{Q_0^2}{(1-(2Q_0)^2)^{1/2}}\left\{\ln\left[\frac{1-2Q_0^2+(1-(2Q_0)^2)^{1/2}}{1-2Q_0^2-(1-(2Q_0)^2)^{1/2}}\right] - \ln\left[\frac{(\Lambda^2-1)2Q_0^2+1+(1-(2Q_0)^2)^{1/2}}{(\Lambda^2-1)2Q_0^2+1-(1-(2Q_0)^2)^{1/2}}\right]\right\}_{0<Q_0<1/2} \tag{15a}$$

$$2\pi K_1(1/2,0) \cong 2(1-1/(\Lambda^2+1)) \tag{15}$$

$$2\pi Q_0 K_1(Q_0,0) = \frac{Q_0^2}{((2Q_0)^2-1)^{1/2}}\, 2\left\{\varphi(Q_0,0)-\varphi(Q_0,\Lambda)\right\}_{1/2<Q_0} \tag{15}$$

$$\tan\varphi(Q_0,\Lambda) = ((2Q_0)^2-1)^{1/2}/((\Lambda^2-1)2Q_0^2+1)$$

$$\frac{\pi}{2} \le \varphi(Q_0,0) \le \pi \quad ; \quad 0 \le \varphi(Q_0,\Lambda)\,.$$

In the limit $Q_0 \longrightarrow \infty$, $K_1(Q_0,0) \longrightarrow K_3(Q_0,0) \longrightarrow 1/2$, one finds the coherent state equality for (11).

Numerical results for $(L\omega_0/\hbar)^{\frac{1}{2}}\sigma(Q,z)$, $(\hbar L\,\omega_0)^{-\frac{1}{2}}\sigma(L\dot{Q},z)$ in (12) and $\epsilon = (\sigma(Q,z)\sigma(L\dot{Q},z) - \hbar/2)/\hbar$ are given in Table I for values of C/C_0 which must be determined experimentally. The parameter ϵ measures the deviation from equality in (11). One sees, in Table I the equal-partition of energy behaviour $K_1(Q_0,z) = K_3(Q_0,z) = z/2$ expected at high temperature and the clearly defined transition region between pure thermal and pure quantum noise.

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CAPTIONS

Fig. 1. The semi-infinite filter.

Table I. Numerical results for the dimensionless second moments and ϵ.

REFERENCES

1. B. Yurke and J.S. Denker, Phys. Rev. A29, 1419 (1984).
2. H. Dekker, Phys. Rep. 80, 1 (1980).
3. J.T. Lewis and H. Maassen, Lecture Notes in Mathematics (Springer-Verlag), Heidelberg 1984) Vol. 1055, pp. 245-276.
4. H. Grabert, U. Weiss and P. Talkner, Z. Phys. B55, 87 (1984).
5. L.D. Landau and E.M. Lifshitz, Electrodynamics of Continuous Media (Pergamon Press, London, 1960) pp. 197-205.
6. T. Garavaglia, Phys. Rev. Lett. 54, 488 (1985).
7. H. Umezawa, H. Matsumoto, and M. Tachiki, Thermo Field Dynamics and Condensed States (North-Holland, Amsterdam, 1982) pps. 119, 144.
8. H.B. Callen and T.A. Welton, Phys. Rev. 83, 34 (1951).
9. L.D. Landau and E.M. Lifshitz, Statistical Physics (Pergamon Press, London, 1980) Part I Ch. 12.

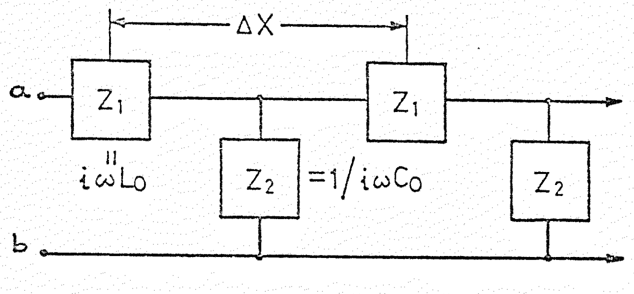

Fig.1.

Table I.

Q_0	z	$\left(\frac{L\omega_0}{\hbar}\right)^{\frac{1}{2}}\sigma(Q,z)$	$\frac{\sigma(L\dot{Q},z)}{(\hbar L\omega_0)^{\frac{1}{2}}}$	ε
		$C/C_0 = 100$		
.25	100	7.209	6.947	49.578
	10	2.283	2.440	5.070
	1	.783	1.757	.875
	0.1	.511	1.743	.390
	.01	.501	1.743	.373
.5	100	7.106	7.048	49.582
	10	2.251	2.433	4.976
	1	.786	1.586	.748
	0.1	.567	1.563	.386
	.01	.567	1.563	.386
1	100	7.080	7.073	49.580
	10	2.242	2.385	4.849
	1	.795	1.373	.592
	0.1	.621	1.336	.330
10	100	7.071	7.076	49.537
	10	.816	2.271	1.354
	1	.696	.845	.088
		$C/C_0 = 10$		
10	100	7.072	7.071	49.509
	10	2.240	2.255	4.551
	1	.808	.888	.218
	0.1	.696	.801	.058
	.01	.696	.801	.058
20	100	7.071	7.072	49.509
	10	2.240	2.250	4.539
	1	.712	.788	.061
	0.1	.702	.763	.035
50	100	7.068	7.069	49.465
	10	2.237	2.244	4.520
	1	.705	.734	.017
100	100	7.036	7.072	49.255
	10	2.130	2.242	4.280
	1	.706	.722	.010